Journal of Statistical Physics, Vol. 127, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10955-007-9327-8

Proof of the Cases $p \leq 7$ of the Lieb-Seiringer Formulation of the Bessis-Moussa-Villani Conjecture

Daniel Hägele[1]

Received February 8, 2007; accepted April 12, 2007
Published Online: May 2, 2007

It is shown that the polynomial $\lambda(t) = \mathrm{Tr}[(A + tB)^p]$ has nonnegative coefficients when $p \leq 7$ and A and B are any two complex positive semidefinite $n \times n$ matrices with arbitrary n. This proves a general nontrivial case of the Lieb-Seiringer formulation of the Bessis-Moussa-Villani conjecture which is a long standing problem in theoretical physics.

KEY WORDS: Bessis-Moussa-Villani conjecture, trace conjecture, partition functions

In 1975, Bessis, Moussa, and Villani (BMV) stated the following, now widely known, conjecture. For arbitrary hermitian matrices G and H the function $t \rightarrow \mathrm{tr}\exp(G + itH)$ is the Fourier transform of a positive measure.(1,2) The conjecture is highly relevant in the context of quantum mechanical partition functions and their derivatives. However, as of today, no complete proof of the conjecture is known, despite "a lot of work, some by prominent mathematical physicists" as Reinhard F. Werner puts it on his homepage.(3)

Recently, Lieb and Seiringer found an equivalent formulation of the BMV conjecture(4) which triggered new attempts to prove it:(5,6)

Conjecture (Bessis-Moussa-Villani) The polynomial $\lambda(t) = \mathrm{Tr}[(A + tB)^p]$ has nonnegative coefficients whenever A and B are n-by-n positive semidefinite matrices.

So far, only the trivial cases for $p \leq 5$ (for references see Ref. 6) along with a single non-trivial case $p = 6$, and matrix dimension $n = 3$ have been proven.(5)

In the following I give an explicit elementary proof for the case $p = 7$ with no restrictions on the size of the matrices A and B. This means that for the first time

[1] Spektroskopie der kondensierten Materie, Ruhr–Universität–Bochum, Universitätsstraße 150, 44780 Bochum, Germany; e-mail: daniel.haegele@ruhr-uni-bochum.de.

0022-4715/07/0600-1167/0

a general non-trivial result on the BMV conjecture for matrix dimensions larger than three could be obtained. According to a theorem of C. J. Hillar (corollary 1.8 in Ref. 6), the proof of $p = 7$ implies that all cases with $p \leq 7$ are proven. Especially the case $p = 6$ and $n = 3$ is covered, which previously had been proven in a completely different way by intricate computer algebra.[5]

The coefficient of t^r in $\lambda(t)$ is the trace of $S_{p,r}(A, B)$, the sum of all words of length p in A and B, in which r Bs appear (sometimes called the r-th Hurwitz product of A and B).

Theorem. *The polynomial $\lambda(t) = \mathrm{Tr}[(A + tB)^p]$ has non-negative coefficients when $p = 7$ and A and B are any two n-by-n positive semidefinite matrices.*

Proof: Consider the third coefficient $\mathrm{Tr}S_{7,3}$. The sum $S_{7,3}$ consists of 35 products where A appears four times and B three times in all possible permutations. Because of the cyclicity of the trace, we can write $\mathrm{Tr}S_{7,3}$ as a sum of only five traces

$$\mathrm{Tr}S_{7,3} = 7(T_1 + T_2 + T_3 + T_4 + T_5), \tag{1}$$

where

$$\begin{aligned}
T_1 &= \mathrm{Tr}(AAAABBB)\\
T_2 &= \mathrm{Tr}(AAABABB)\\
T_3 &= \mathrm{Tr}(AAABBAB)\\
T_4 &= \mathrm{Tr}(AABAABB)\\
T_5 &= \mathrm{Tr}(AABABAB).
\end{aligned}$$

To prove that $\mathrm{Tr}S_{7,3}$ is nonnegative I will show that it can be written as a sum of traces of the form $Q = \mathrm{Tr}C^*C$, where C is a suitably chosen complex matrices. The relation $\mathrm{Tr}C^*C \geq 0$ holds for any C because any matrix of the form C^*C is positive semidefinite. Consider

$$C = c_1 bAAB + c_2 bABA + c_3 bBAA,$$

where the c_js are complex coefficients and a and b are hermitian n-by-n matrices with the properties $a^2 = A$ and $b^2 = B$. The matrices a and b exist because A and B are positive semidefinite and therefore also hermitian. We obtain

$$\begin{aligned}
Q(c_1, c_2, c_3) &= \mathrm{Tr}CC^*\\
&= \mathrm{Tr}(c_1c_1^* bAABBAAb + c_1c_2^* bAABABAb + c_1c_3^* bAABAABb\\
&\quad + c_2c_1^* bABABAAb + c_2c_2^* bABAABAb + c_2c_3^* bABAAABb\\
&\quad + c_3c_1^* bBAABAAb + c_3c_2^* bBAAABAb + c_3c_3^* bBAAAABb)\\
&= c_1c_1 T_4 + c_1c_2^* T_5 + c_1c_3^* T_4
\end{aligned}$$

$$+ c_2 c_1^* T_5 + c_2 c_2^* T_5 + c_2 c_3^* T_3$$
$$+ c_3 c_1^* T_4 + c_3 c_2^* T_2 + c_3 c_3^* T_1 \geq 0.$$

By comparison with (1) we find

$$\mathrm{Tr} S_{7,3} = 7Q(1,0,0) + 7Q(0,1,1) \geq 0,$$

which concludes the proof of the third coefficient being nonnegative.

Similarly to the proof above, the nonnegativity of the 0th, 1st and 2nd coefficient can be proven:

$$\begin{aligned}
\mathrm{Tr} S_{7,2} &= 7\mathrm{Tr}(BAAAAAB + ABAAABA + AABABAA) \\
&= 7\mathrm{Tr}(BAAa * H.C. + ABAa * H.C. + AABa * H.C.) \geq 0, \\
\mathrm{Tr} S_{7,1} &= 7\mathrm{Tr}(AAABAAA) \\
&= 7\mathrm{Tr}(AAAb * H.C.) \geq 0, \\
\mathrm{Tr} S_{7,0} &= \mathrm{Tr}(AAAAAAA) \\
&= \mathrm{Tr}(AAAa * H.C.) \geq 0,
\end{aligned}$$

where H.C. denotes the hermitian conjugate of the first factor in the product. The cases $\mathrm{Tr}S_{7,4}$,$\mathrm{Tr}S_{7,5}$,$\mathrm{Tr}S_{7,6}$ and $S_{7,7}$ are, after exchanging A and B, identical with the cases $\mathrm{Tr}S_{7,3}$,$\mathrm{Tr}S_{7,2}$, $\mathrm{Tr}S_{7,1}$, and $S_{7,0}$, respectively and are therefore covered by the proof above. This completes the proof of the theorem. □

Corollary. *The polynomial $\lambda(t) = \mathrm{Tr}[(A + tB)^p]$ has non-negative coefficients when $p \leq 7$ and A and B are any two n-by-n positive semidefinite matrices.*

Proof: The corollary follows immediately from a theorem of C. J. Hillar.

Theorem. *(Corollary 1.8 in Ref. 6). If the Bessis-Moussa-Villani conjecture (see above) is true for some p_0, then it is also true for all $p < p_0$.* □

The case $p = 6$. One may ask if it is possible to find a direct way of proving the case $p = 6$ in a fashion similar to the proof shown above. This is not possible as I will show below. Analogously to the case $p = 7$, we find after some algebra

$$\mathrm{Tr} S_{6,3} = 6T_1 + 6T_2 + 6T_3 + 2T_4. \qquad (2)$$

where

$$T_1 = \mathrm{Tr}(AAABBB)$$
$$T_2 = \mathrm{Tr}(AABABB)$$

$$T_3 = \mathrm{Tr}(AABBAB)$$
$$T_4 = \mathrm{Tr}(ABABAB).$$

Using

$$C = c_1\, aABb + c_2\, aBAb$$

we obtain

$$\begin{aligned} Q(c_1, c_2) &= \mathrm{Tr}CC^* \\ &= \mathrm{Tr}(c_1 c_1^* aABbbBAa + c_1 c_2^* aABbbABa \\ &\quad \times c_2 c_1^* aBAbbBAa + c_2 c_2^* aBAbbABa) \geq 0. \end{aligned}$$

and

$$Q(c_1, c_2) = c_1 c_1^* T_1 + c_1 c_2^* T_3 + c_2 c_1^* T_2 + c_2 c_2^* T_4 \geq 0.$$

Here it is impossible to find a way to write $\mathrm{Tr}S_{6,3}$ as a sum of $Q(c_1, c_2)$: Suppose there is a solution, then from

$$\begin{aligned} \mathrm{Tr}S_{6,3} &= \sum_l Q(c_{1,l}, c_{2,l}) \\ &= \sum_l c_{1,l} c_{1,l}^* T_1 + \sum_l c_{1,l} c_{2,l}^* T_3 + \sum_l c_{2,l} c_{1,l}^* T_2 + \sum_l c_{2,l} c_{2,l}^* T_4 \end{aligned}$$

we find by comparison with Eqn. (2) $\sum_j c_{1,l} c_{1,l} = 6$, $\sum_l c_{1,l} c_{2,l}^* = 6$, $\sum_l c_{2,l} c_{1,l}^* = 6$, and $\sum_l c_{2,l} c_{2,l}^* = 2$. Since generally $(c_{1,l} - c_{2,l})(c_{1,l} - c_{2,l})^* \geq 0$ we find

$$\sum_l c_{1,l} c_{1,l}^* + \sum_l c_{2,l} c_{2,l}^* \geq \sum_l c_{1,l} c_{2,l}^* + \sum_l c_{2,l} c_{1,l}^*$$
$$6 + 3 \geq 6 + 6,$$

which contradicts our assumption that $\mathrm{Tr}S_{6,3}$ can be written as sum of Qs. Note that the C was chosen to yield the right number of As and Bs in the sum. We speculate that a more general ansatz that relaxes this restriction for C, i.e. $C_1 = c_1\, aABb + c_2\, aBAb + c_3\, aAAb + c_4\, aBBb$ and $C_2 = d_1 aAAa + d_2 aABa + \ldots$, $C_3 = \ldots, \ldots$ may lead to a direct proof.

In conclusion, I gave an elementary proof of the nontrivial case $p = 7$ of the Lieb-Seiringer formulation of the Bessis-Moussa-Villani conjecture for $n \times n$-matrices. Although a proof of $p = 7$ implies the $p = 6$ case, the way of proving the $p = 7$ case did not apply analogously. Attempts to prove cases with $p > 7$ by methods similar to those presented here will be more complex, but should nevertheless be undertaken in the future using computer algebra.

REFERENCES

1. D. Bessis, P. Moussa and M. Villani, Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *J. Math. Phys.* **16**:2318–2325 (1975).
2. P. Moussa, On the Representation of Tr($e^{A-\lambda B}$) as a Laplace Transform. *Rev. Math. Phys.* **12**:621–655 (2000).
3. http://www.imaph.tu-bs.de/home/werner/problems.html.
4. E. H. Lieb and R. Seiringer, Equivalent forms of the Bessis-Moussa-Villani conjecture. *J. Stat. Phys.* **115**:185–190 (2004).
5. C. J. Hillar and C. R. Johnson, On the positivity of the coefficients of a certain polynomial defined by two positive definite matrices. *J. Stat. Phys.* **118**:781–789 (2005).
6. C. J. Hillar, Advances on the Bessis-Moussa-Villani trace conjecture, math.OA/0507166v4 (2005).